ON THE ABSENCE OF STABILITY OF BASES IN SOME FRÉCHET SPACES.

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ABSTRACT. We show that, for each compact subset of the real line of infinite cardinality with an isolated point, the space of Whitney jets on the set does not possess a basis consisting only of polynomials. On the other hand, polynomials are dense in any Whitney space. Thus, there are no general results about stability of bases in Fréchet spaces.

Dedicated to the memory of M. M. Dragilev

1. Introduction

Let X be a Banach space with a Schauder basis. By the Krejn-Mil'man-Rutman theorem (see Theorem 3.2 below) the basis is stable. This means that sufficiently small perturbations of basis elements preserve the basis property of the system. Hence, if X is a function space such that polynomials are contained and dense in X, then the space possesses a basis consisting only of polynomials. Our aim is to show that, for Fréchet spaces, the situation may differ. Let K be a compact subset of $\mathbb{R}$ containing infinitely many points and such that the set of isolated points is not empty. We show that the space of Whitney jets $\mathscr{E}(K)$ cannot have a basis of polynomials. Clearly, polynomials are dense in each Whitney space. Combining these facts, we see that *there are no general conditions for stability of bases in Fréchet spaces.*

The paper is organized as follows. Section 2 contains the main result about the absence of polynomial bases in some Whitney spaces. In Section 3, we recall known results about stability of bases in Banach spaces and their generalizations to the case of Fréchet spaces. Thus, we get an apparent contradiction of our result with the theorems on stability of bases in Fréchet spaces. To clarify this seeming contradiction, by way of illustration, we consider in Section 4 an example of a set K from the considered class with a known basis of the space $\mathscr{E}(K)$. For the set $K=[-1,1]\cup\{2\}$ we present a basis in the space $\mathscr{E}(K)$ and analyze the conditions of proximity in the stability theorems for Fréchet spaces. We show that, in our case, these conditions cannot be achieved even though for elements from a dense set.

At the end of the article, a hypothesis on the form of bases in Whitney spaces is proposed.

2. The absence of polynomial bases in some Whitney spaces

Let X be a linear topological space over the field $\mathbb{K}$. A sequence $(e_n)_{n=1}^\infty \subset X$ is a (topological) basis for X if for each $f\in X$ there is a unique sequence $(\xi_n(f))_{n=1}^\infty \subset \mathbb{K}$ such that the series $\sum_{n=1}^\infty \xi_n(f)\,e_n$ converges to f in the topology of X. In the case of Fréchet spaces, the functionals ξ_n are continuous, so $(e_n)_{n=1}^\infty$ is a Schauder basis.

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We consider bases in the Whitney spaces. Let K be a compact subset of $\mathbb{R}$ and I be a closed interval containing K. The Whitney space $\mathscr{E}(K)$ consists of traces on K of functions from $C^\infty(I)$, that is the element f of $\mathscr{E}(K)$ is a jet $(f^{(j)}(x))_{x\in K,\, j\geq 0}$ such that there exists an extension $F\in C^\infty(I)$ with $F^{(j)}(x)=f^{(j)}(x)$ for all $x\in K$ and $j\in\mathbb{Z}_+$.

Since $\mathscr{E}(K)$ is a factor space, it should be equipped with the quotient topology. By Whitney [16], this topology is given by the seminorms

$$\|f\|_q = |f|_{q,K} + \sup\left\{|(R^q_y f)^{(n)}(x)|\cdot|x-y|^{n-q} : x\neq y,\, n\leq q\right\},\ \ q\in\mathbb{Z}_+. \tag{2.1}$$

Here, $|f|_{q,K}=\sup\{|f^{(k)}(x)| : x\in K, k\leq q\}$ and $R^q_y f(\cdot)=f(\cdot)-\sum_{k=0}^q \frac{f^{(k)}(y)}{k!}(\cdot - y)^k$ is the $q-$th Taylor remainder of f at y.

In the case of a singleton $\{a\}$, due to Borel (see e.g. [9], p.69), $\mathscr{E}(\{a\})\simeq\omega=\mathbb{R}^{\mathbb{N}}$. Its dual is the space φ of finite sequences, see e.g. [7], p.288.

The natural basis of the space $\mathscr{E}(\{a\})$ is given by the sequence $(e_n)_{n=0}^\infty$, where the jet e_n is defined by the function $e_n(x)=\frac{(x-a)^n}{n!}$ at $x=a$, so $e_n^{(j)}(x)=1$ if $j=n$ and $x=a$ and $e_n^{(j)}(x)=0$ for all other $x\in K$ and $j\in\mathbb{Z}_+$. The functionals $\eta_n(f)=f^{(n)}(a)$ with $n\geq 0$ are biorthogonal to $(e_n)_{n=0}^\infty$, that is $\eta_n(e_m)=\delta_{nm}$. Since the sequence $(\eta_n)_{n=0}^\infty$ forms a basis in the dual space, for any functional $\eta\in\mathscr{E}(\{a\})'$ there exist $\beta_0,\dots,\beta_N$ such that

$$\eta(f)=\sum_{k=0}^N \beta_k f^{(k)}(a),\ \ f\in\mathscr{E}(\{a\}). \tag{2.2}$$

Suppose K contains infinitely many points and the set of isolated points of K is not empty. Then $K=K_0\cup\{a\}$, where the point a is isolated. This representation yields a decomposition $\mathscr{E}(K)=\mathscr{E}(K_0)\oplus\mathscr{E}(\{a\})$ and, correspondingly, for the dual spaces

$$\mathscr{E}(K)'=\mathscr{E}(K_0)'\oplus\mathscr{E}(\{a\})'. \tag{2.3}$$

Given a function $f\in\mathscr{E}(K)$, let $F\in C^\infty(I)$ be any extension of f. Then, for $n\leq q$, by the Lagrange form of the remainder, $(R^q_y f)^{(n)}(x)=[F^{(q)}(\theta)-f^{(q)}(y)]\,(x-y)^{q-n}/(q-n)!$ for some point θ between x and y. By (2.1), this gives $\|f\|_q\leq 3\,|F|_{q,I}$.

It is easily seen that polynomials are dense in each Whitney space. Indeed, without loss of generality we can assume that $K\subset[-1,1]$. Then, given $f\in\mathscr{E}(K)$ and $q\in\mathbb{Z}_+$, by e.g. Theorem A in [6], we can approximate any extension $F\in C^\infty([-1,1])$ together with all its derivatives up to order q. Hence, for each $\varepsilon>0$ there is a polynomial P with $|F-P|_{q,[-1,1]}<\varepsilon$. Therefore, $||f-P||_q<3\,\varepsilon$. The base of neighborhoods of zero in the space $\mathscr{E}(K)$ is given by the sets $U_{q,m}=\{g\in\mathscr{E}(K) : ||g||_q<\frac{1}{m}\}$ for $q\in\mathbb{Z}_+$ and $m\in\mathbb{N}$. Thus, each neighborhood of f contains a polynomial.

Suppose the space $\mathscr{E}(K)$ has a topological basis $(f_n)_{n=1}^\infty$ with the corresponding biorthogonal functionals $(\zeta_n)_{n=1}^\infty$. Since the space is nuclear, by the Dynin-Mityagin theorem ([12], T.9), the basis is absolute. Hence, for each $q\in\mathbb{Z}_+$ and for each $f\in\mathscr{E}(K)$ the series

$$\sum_{n=1}^\infty |\zeta_n(f)|\cdot||f_n||_q \tag{2.4}$$

converges.

It should be noted that absoluteness of bases in some spaces of analytic functions was originally proved by M. M. Dragilev in [3].

Theorem 2.1. *Let $K \subset \mathbb{R}$ be an infinite compact set with an isolated point. Then the space $\mathscr{E}(K)$ does not possess a polynomial basis.*

Proof. Suppose, to derive a contradiction, that the sequence of polynomials $(f_n)_{n=1}^{\infty}$ with the biorthogonal functionals $(\zeta_n)_{n=1}^{\infty}$ forms a basis in $\mathscr{E}(K)$. As above, let $K = K_0 \cup \{a\}$, where a is an isolated point.

Take $q = 0$ in (2.4). Since f_n is a polynomial and the set K is infinite, the value $\varepsilon_n := ||f_n||_0$ is positive for each n. By (2.3), $\zeta_n = \xi_n + \eta_n$ where $\xi_n \in \mathscr{E}(K_0)'$ and $\eta_n \in \varphi$. By (2.2), for each n there is a finite set $(\beta_{k,n})_{k=0}^{N_n}$ with $\beta_{N_n,n} \neq 0$ so that

$$\eta_n(f) = \sum_{k=0}^{N_n} \beta_{k,n} f^{(k)}(a)$$

for each $f \in \mathscr{E}(K)$. In particular, if $f|_{K_0} = 0$ then $\xi_n(f) = 0$ for all n. Therefore, for such functions, the series $\sum_{n=1}^{\infty} |\eta_n(f)| \cdot \varepsilon_n$ converges. The functionals $(\eta_n)_{n=1}^{\infty}$ are linearly independent, so the sequence $(N_n)_{n=1}^{\infty}$ is not bounded. Choose $(n_j)_{j=1}^{\infty}$ such that N_{n_j} increases strictly. Then the series $\sum_{j=1}^{\infty} |\eta_{n_j}(f)| \cdot \varepsilon_{n_j}$ converges for each f with $f|_{K_0} = 0$, which is impossible, since by induction one can choose $f^{(N_{n_j})}(a)$ large enough such that $|\eta_{n_j}(f)| \geq \varepsilon_{n_j}^{-1}$ for all j. □

Remark. There might be a generalization of the theorem to the case of non-algebraic compact set $K \subset \mathbb{R}^N$ with an isolated point.

Corollary 2.2. *There are no general conditions for stability of bases in Fréchet spaces.*

Indeed, we expect from such conditions a possibility to apply them at least to elements from a dense subset.

Nevertheless there are two theorems on stability of bases in Fréchet spaces. We consider them in the next section.

3. **Arsove generalization of the Paley-Wiener Theorem**

For the convenience of the reader, first we recall two theorems about stability of bases in Banach spaces.

Theorem 3.1. (Paley-Wiener Theorem) *Let $(f_n)_{n=1}^{\infty}$ be a basis for a Banach space X and $(g_n)_{n=1}^{\infty}$ be vectors in X. Suppose there exists a constant $\lambda \in [0,1)$ such that the inequality*

$$||\sum_{n=1}^{N} c_n(f_n - g_n)|| \leq \lambda ||\sum_{n=1}^{N} c_n f_n||$$

holds for all finite sequences $c_1, c_2, \ldots, c_N$ of scalars. Then $(g_n)_{n=1}^{\infty}$ is a basis for X.

The theorem was proved in [14] for Hilbert spaces, see also [9], p.163. Its extension to the case of Banach spaces was given in [2], Theorem 1.1.

Theorem 3.2. (Krejn-Mil'man-Rutman Theorem)[8] *Let $(f_n)_{n=1}^\infty$ be a basis for a Banach space X and $(\zeta_n)_{n=1}^\infty$ be a sequence of biorthogonal functionals. Then each system of vectors $(g_n)_{n=1}^\infty$ satisfying the condition*

$$\sum_{n=1}^{\infty} ||\zeta_n|| \cdot ||f_n - g_n|| < 1$$

is a basis in X.

Corollary 3.3. *Suppose a Banach function space X has a basis and polynomials are dense in X. Then X possesses a polynomial basis.*

Indeed, given a biorthogonal system $(f_n, \zeta_n)_{n=1}^\infty$, for each n we can choose a polynomial g_n with $||f_n - g_n|| < 2^{-n-1}\,||\zeta_n||^{-1}$.

The next generalizations of the Paley-Wiener theorem for the case of Fréchet spaces are due to Arsove [1], see also Theorem IX.4.4 in [10].

Assume that X is a Fréchet space whose topology is given by an increasing family of seminorms $(||\cdot||_q)_{q=0}^\infty$.

Theorem 3.4. ([1], T.5) *Let $(f_n)_{n=1}^\infty$ be a basis for X and $(g_n)_{n=1}^\infty$ be vectors in X. Suppose there exists a sequence $(\lambda_q)_{q=0}^\infty$ with $\lambda_q \in [0,1)$ such that the inequality*

$$||\sum_{n=1}^{N} c_n(f_n - g_n)||_q \le \lambda_q\, ||\sum_{n=1}^{N} c_n f_n||_q$$

holds for all $q \in \mathbb{Z}_+$ and all finite sequences $c_1, c_2, \dots, c_N$ of scalars. Then $(g_n)_{n=1}^\infty$ is a basis for X.

The Fréchet metric of X is given as $\rho(f,g) = \sum_{q=0}^\infty 2^{-q-1} \frac{||f-g||_q}{1+||f-g||_q}$. We can consider as well the second version of the generalized Paley-Wiener theorem.

Theorem 3.5. ([1], T.1) *Let $(f_n)_{n=1}^\infty$ be a basis for X and $(g_n)_{n=1}^\infty$ be vectors in X. Suppose there exists a constant $\lambda \in [0,1)$ such that*

$$\rho(\sum_{n=1}^{N} c_n(f_n - g_n), 0) \le \lambda \cdot \rho(\sum_{n=1}^{N} c_n f_n, 0)$$

holds for all finite sequences $c_1, c_2, \dots, c_N$ of scalars. Then $(g_n)_{n=1}^\infty$ is a basis for X.

4. **Example and conjecture**

We see that Corollary 3.3 cannot be extended to the case of all Fréchet spaces. In order to illustrate why the last two theorems do not imply the existence of polynomial bases in the spaces of Whitney jets, we analyze the proximity conditions in these theorems. As example, we consider the simplest infinite compact set K of the considered class with a known basis of the space $\mathscr{E}(K)$.

Example. Let $K = [-1,1] \cup \{2\}$. Then $\mathscr{E}(K) = \mathscr{E}([-1,1]) \oplus \mathscr{E}(\{2\})$. If $X = Y \oplus Z$ and bases $(y_n)_{n=0}^\infty$, $(z_n)_{n=0}^\infty$ of the spaces Y, Z, respectively, are given, then, clearly, the sequence $y_0, z_0,\ y_1, z_1, \dots, y_n, z_n, \dots$ is a basis in the space X. In our case, by Lemma 25 in [12], the Chebyshev polynomials $(T_n)_{n=0}^\infty$ form a basis in the space $\mathscr{E}([-1,1])$. For a basis in the space $\mathscr{E}(\{2\})$ we take the jets $(e_n)_{n=0}^{\infty}$ given by the functions $(x-2)^n/n!$ at $x=2$, $n \in \mathbb{Z}_+$. Let $f_{2n-1}(x) = T_{n-1}(x)$ for $|x| \le 1, f_{2n-1}^{(j)}(2) = 0$ and $f_{2n} = e_{n-1}$ for $n \in \mathbb{N}, j \in \mathbb{Z}_+$. Then $(f_n)_{n=1}^{\infty}$ is a basis in $\mathscr{E}(K)$.

Let us illustrate why, for each sequence of polynomials $(g_n)_{n=1}^{\infty}$ and $(\lambda_q)_{q=0}^{\infty} \subset [0,1)$, the condition of Theorem 3.4 cannot be satisfied. Suppose, for contradiction, such sequences exist. Let us take $f_4 = e_1$, so this is a jet given by the function $x-2$ at $x=2$. An easy computation gives $||f_4||_0 = 0$ and $||f_4||_q = 2$ for $q \ge 1$. We consider the polynomial g_4 that corresponds to f_4 in the sense of the inequality in Theorem 3.4. Take $c_k = 0$ for all k, except $c_4 = 1$. Then, for $q = 0$ we have $||f_4 - g_4||_0 = ||g_4||_0 \le \lambda_0 ||f_4||_0 = 0$. Therefore, $g_4 = 0$, which is impossible, because, in this case, for $q = 1$ we get $||f_4 - g_4||_1 = ||f_4||_1 \le \lambda_1 ||f_4||_1$, a contradiction for $\lambda_1 < 1$ and $||f_4||_1 = 2$.

In the case of Theorem 3.5, similarly, given polynomial g_4 and $0 < \lambda < 1$, we see that $g_4 \ne 0$. Let $c_k = 0$ for all k, except $c_4 = M$ for large positive M. Then the condition of the theorem has the form

$$\rho(M(f_4 - g_4), 0) \le \lambda \cdot \rho(M f_4, 0). \tag{4.1}$$

For the known values of $||f_4||_q$ with $q \in \mathbb{Z}_+$, we have $\rho(M f_4, 0) = \frac{1}{2} \frac{2M}{1+2M}$. Hence the right-hand side of (4.1) does not exceed $\frac{\lambda}{2}$. On the other hand, we can estimate $\rho(M(f_4 - g_4), 0)$ from below by means of only term of the sum corresponding to $q = 0$. Hence the left-hand side of (4.1) exceeds $\frac{1}{2} \frac{M||g_4||_0}{1+M||g_4||_0}$. Since g_4 is a nontrivial polynomial, the value $||g_4||_0$ is positive. For large enough values M, this fraction is so closed to $\frac{1}{2}$, as we wish, so it exceeds $\frac{\lambda}{2}$, a contradiction.

We see that Theorems 3.4, 3.5 have somewhat limited applicability, since the proximity conditions in them are too strong for some Fréchet spaces.

The existence of polynomial bases in a Whitney space $\mathscr{E}(K)$ is not related with the extension property of the set K (availability of a continuous linear extension operator $W : \mathscr{E}(K) \to C^\infty(I)$ or, equivalently, the dominating norm property of the space $\mathscr{E}(K)$, see e.g. [11] for the definition of the DN property). In [4], bases were constructed for Cantor type sets $K(\Lambda)$. Choosing a Cantor type set with fast decreasing lengths of intervals in the Cantor procedure, we can get a space $\mathscr{E}(K(\Lambda))$ without DN property. In addition, both for small sets $K(\Lambda)$ and for the set $[-1,1]$, Faber bases were presented in [4] and [5]. In both cases, bases were given by means of the Newton interpolating polynomials with nodes at “nearly” Leja points.

Recall that a polynomial basis $(P_n)_{n=0}^{\infty}$ in a function space is called a Faber (or strict polynomial) basis if $\deg P_n = n$ for all n, see e.g. [13]. Also, points $(a_k)_{k=1}^{\infty} \subset K$ are Leja if $a_1 \in K$ is arbitrary, and, once $a_1, a_2, \dots, a_{k-1}$ have been determined, a_k is chosen so that it provides the maximum modulus of the polynomial $(x-a_1)\cdots(x-a_{k-1})$ on K. For applications of Leja points in Approximation Theory we refer the reader to [15].

Based on these considerations, we put forward the following hypothesis.

Conjecture. Given a compact set $K \subset \mathbb{R}$ of infinite cardinality, the space $\mathscr{E}(K)$ has a polynomial basis if and only if the set K is perfect. In addition, if K is perfect, then $\mathscr{E}(K)$ possesses a strict polynomial basis.

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